# Random continuum and Brownian motion

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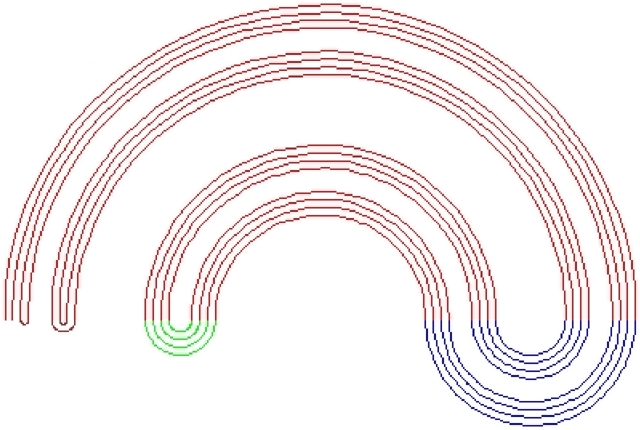
# Continua

A **continuum** is a metric compact connected space.

A continuum is **indecomposable** if it is not the union of two proper subcontinua.

A continuum $C$ is **chainable** if, for each $\epsilon > 0$, there exists a continuous $f\colon C \to [0,1]$ such that $\mathrm{diam}\big(f^{-1}(r)\big) < \epsilon$.

# Knaster's continuum



# A real-life picture of Knaster's continuum

A continuum is **hereditarily indecomposable** if each of its subcontinua is indecomposable.

**Knaster**, 1922: There exists a non-degenerate chainable hereditarily indecomposable continuum.

**Bing**, 1951: All non-degenerate chainable hereditarily indecomposable continua are homeomorphic to each other (**the pseudoarc**).

# Sequences of functions determining a continuum

A **suitable interval** is a compact interval in $\mathbb{R}$ containing 0.

$\bar{f} = (f_n)$ a sequence of continuous functions $\mathbb{R} \to \mathbb{R}$ with $f_n(0) = 0$

$J$ be a non-degenerate suitable interval

Consider the following matrix of intervals

$$\begin{array}{cccc} f_1(J), & f_1 \circ f_2(J), & f_1 \circ f_2 \circ f_3(J), & f_1 \circ f_2 \circ f_3 \circ f_4(J), \ \dots \\ & f_2(J), & f_2 \circ f_3(J), & f_2 \circ f_3 \circ f_4(J), \ \dots \\ & & f_3(J), & f_3 \circ f_4(J), \ \dots \\ & & \vdots & \end{array}$$

If each row has a **limit** (with respect to the Hausdorff metric on intervals) that is **independent of** $J$, then we say that $\bar{f}$ **determines a continuum**.

$\bar{f}$ **determines a continuum** if there exists a sequence $(I_n)$ of suitable intervals such that, for each non-degenerate suitable interval $J$,

$$I_n = \lim_k \big(f_n \circ \cdots \circ f_{n+k}\big)(J).$$

Note that the sequence $(I_n)$ is determined solely by $\bar{f}$.

Observe that $I_n = f_n(I_{n+1})$, for each $n \in \mathbb{N}$.
Define the inverse limit

$$K(\bar{f}) = \varprojlim_n (I_n, f_n \restriction I_{n+1}).$$

$K(\bar{f})$ is a chainable continuum. We call it the **continuum determined by** $\bar{f}$.

**Examples.**

The sequence $\bar{f}$ with $f_n = \mathrm{id}_{\mathbb{R}}$ does not determine a continuum.

The sequence $\overline{g}$ with $g_n(t) = \sin(\pi n t)$ does determine a continuum.

# Brownian motion

**Wiener**, 1923: A **Brownian motion** is a measurable function

$$B\colon \Omega \times \mathbb{R}^+ \to \mathbb{R},$$

where $\Omega$ is a probability space, such that

— $B(\cdot, 0) = 0$;

— for $0 \le t_0 < \cdots < t_k$, the random variables $B(\cdot, t_i) - B(\cdot, t_{i-1})$, $i = 1, \ldots, k$, are independent;

— for $0 \le s < t$, $B(\cdot, t) - B(\cdot, s)$ is a normally distributed random variable with expectation 0 and variance $t - s$;

— the function $B(\omega, \cdot)$ is continuous, on a measure 1 set of $\omega \in \Omega$.

We write $B(t)$ for $B(\omega, t)$.

**Paley, Wiener, Zygmund**, 1933 (**Perrin**, 1914): $B(t)$ is **not** differentiable at each $t$, **almost surely**.

**Hermite**, 1893: I turn with terror and horror from this lamentable scourge of continuous functions with no derivatives.

Let

$$B_+(t) \text{ and } B_-(t), \text{ for } t \geq 0,$$

be two independent Brownian motions.
$B$ defined by

$$B(t) = B_+(t), \text{ if } t \geq 0, \quad \text{and} \quad B(t) = B_-(-t), \text{ if } t < 0,$$

is called a **two-sided Brownian motion**.

Our main object is

$$(B_n)_{n \in \mathbb{N}}$$

a **sequence of independent two-sided Brownian motions**.

# Theorem

## Theorem (Kiss–Solecki, 2020)

(i) *The sequence $\bar{B} = (B_n)_n$ of independent two-sided Brownian motions determines a non-degenerate continuum with probability 1.*

(ii) *The continuum $K(\bar{B})$ determined by $\bar{B}$ is indecomposable with probability 1.*

Making explicit the dependence on $\omega$ from the probability space $\Omega$, Theorem asserts that **on a set of measure 1 of** $\omega \in \Omega$

the function $B_n(\omega, \cdot)$ is continuous for each $n \in \mathbb{N}$,

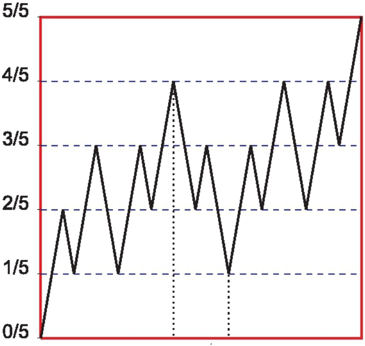
the sequence $(B_n(\omega, \cdot))_n$ determines a non-degenerate continuum,

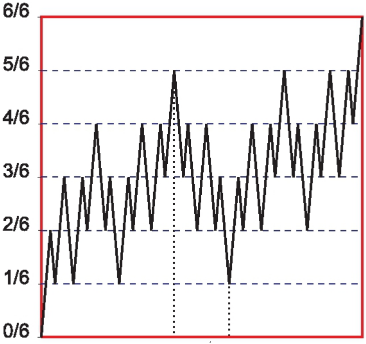
the continuum determined by $(B_n(\omega, \cdot))_n$ is indecomposable.

# Background

**Bing**, *The pseudo-arc*, in *Summer Institute on Set Theoretic Topology*, Madison, 1955.

A frustrating walk defining the pseudo-arc



**Prajs**, *Open problems in the study of homogeneous continua*, plenary talk, 52nd Spring Topology and Dynamical Systems Conference, Auburn University, 2018.

**Levine**, *Conversations*

**Curien**, **Konstantopoulos**, *Iterating Brownian motions, ad libitum*, J. Theoret. Probab., 2014.

# About the proof

## About the proof of (i).

The point is to find, almost surely, a sequence $(I_n)$ of non-degenerate suitable intervals such that, for each non-degenerate suitable interval $J$, we have

$$
\begin{array}{rrrrl}
B_1(J), & B_1 \circ B_2(J), & B_1 \circ B_2 \circ B_3(J), & B_1 \circ B_2 \circ B_3 \circ B_4(J), & \cdots \to \mathbf{I_1} \\
 & B_2(J), & B_2 \circ B_3(J), & B_2 \circ B_3 \circ B_4(J), & \cdots \to \mathbf{I_2} \\
 & & B_3(J), & B_3 \circ B_4(J), & \cdots \to \mathbf{I_3} \\
 & & \vdots & &
\end{array}
$$

Define

$$W_k(t) = (B_1 \circ B_2 \circ \cdots \circ B_k)(t).$$

For $I$ an interval and $c > 0$, let
$c * I =$ the interval with the same center as $I$ and $|c * I| = c|I|$

## Lemma

*For every $\varepsilon > 0$ and $\delta > 0$, there exists $k$ such that*

$$\mathbb{P}\left((1+\varepsilon) * W_k([-\delta, \delta]) \supseteq W_k\left(\left[-\frac{1}{\delta}, \frac{1}{\delta}\right]\right)\right) > 1 - \varepsilon.$$

Let

$$\Delta_k(t) = |W_k([0,t])|,$$

and let $D_1, D_2, \ldots$ be independent copies of $\Delta_1(1)$.

Then by Curien–Konstantopoulos

$$\Delta_k(t) \text{ and } t^{2^{-k}} \prod_{i=1}^{k} D_i^{2^{-(i-1)}}$$

have the same distribution.

This leads to

$$\mathbb{E}\log\big(|W_k([-1/\delta, 1/\delta])|\big) - \mathbb{E}\log\big(|W_k([-\delta,\delta])|\big) = \frac{\log(1/\delta^2)}{2^k}.$$

## Lemma

*For every* $\varepsilon > 0$*, there exists* $1 > \delta > 0$ *such that, for* $k \in \mathbb{N}$*,*

$$\mathbb{P}\left(\forall n \in \mathbb{N}\left(B_{k+n}\left(\left[\frac{-2^n}{\delta}, \frac{2^n}{\delta}\right]\right) \subseteq \left[\frac{-2^{n-1}}{\delta}, \frac{2^{n-1}}{\delta}\right]\right)\right) > 1 - \varepsilon.$$

$$\mathbb{P}\left(\forall n \in \mathbb{N}\left(B_{k+n}\left(\left[0, \frac{\delta}{2^n}\right]\right) \supseteq \left[\frac{-\delta}{2^{n-1}}, \frac{\delta}{2^{n-1}}\right]\right)\right) > 1 - \varepsilon,$$

**Assume the lemmas.** Fix $\epsilon > 0$. The second lemma gives $\delta = \delta(\epsilon) > 0$.

$\kappa =$ the $k$ chosen in the first lemma for $\varepsilon$ and $\delta$

Set

$$I(\varepsilon) = W_\kappa([-\delta, \delta]).$$

With probability greater than $1 - 4\varepsilon$, for each $n$,

$$W_\kappa \circ B_{\kappa+1} \circ \cdots \circ B_{\kappa+n}\left(\left[\frac{-2^n}{\delta}, \frac{2^n}{\delta}\right]\right) \subseteq W_\kappa\left(\left[\frac{-1}{\delta}, \frac{1}{\delta}\right]\right)$$

$$\subseteq (1+\varepsilon) * I(\varepsilon),$$

$$W_\kappa \circ B_{\kappa+1} \circ \cdots \circ B_{\kappa+n}\left(\left[0, \frac{\delta}{2^n}\right]\right) \supseteq I(\varepsilon).$$

For a suitable interval $J$, for large $n$,

$$J \subseteq [-2^n/\delta, 2^n/\delta] \text{ and either } [0, \delta/2^n] \subseteq J \text{ or } [-\delta/2^n, 0] \subseteq J.$$

So, with probability greater than $1 - 4\varepsilon$,

$$I(\varepsilon) \subseteq W_{\kappa+n}(J) \subseteq (1+\varepsilon) * I(\varepsilon),$$

for all suitable $J$ and large enough $n$ depending on $J$.

## About the proof of (ii).

For a suitable interval $I$, let

$$w(I) = \min\big(\max(I), -\min(I)\big).$$

$(I_n)$ = the sequence of suitable intervals found in (i) of the theorem

**Lemma**

$\limsup_n w(I_n) > 0$ *with probability 1.*

Set

$$a_n = w(I_n).$$

Need: $\limsup_n a_n > 0$.

There exists $d > 0$ such that, for small $\epsilon > 0$,

$$\mathbb{P}(a_1 < d) < 1 - 2\epsilon.$$

Since the sequence $(a_n)$ is identically distributed, for small $\epsilon > 0$,

$$\forall n \ \mathbb{P}(a_n < d) < 1 - 2\epsilon.$$

Find a sequence $1 = n_0 < n_1 < n_2 < \cdots$ such that

$$\mathbb{P}\Big(\mathrm{dist}\big(I_{n_k}, B_{n_k} \circ \cdots \circ B_{n_{k+1}-1}([0,1])\big) < \frac{d}{3}\Big) > 1 - \frac{\epsilon}{2^{k+1}}.$$

Define

$$b_k = w\big(B_{n_k} \circ \cdots \circ B_{n_{k+1}-1}([0,1])\big).$$

$(b_k)$ is a sequence of independent random variables.

Recall

$$a_{n_k} = w(I_{n_k}).$$

Comparing $(a_{n_k})$ and $(b_k)$:

$$\mathbb{P}\big(\forall k \; a_{n_k} > b_k - \frac{d}{3}\big) > 1 - \sum_k \frac{\epsilon}{2^{k+1}} = 1 - \epsilon$$

$$\mathbb{P}\big(b_k < \frac{2d}{3}\big) \leq \mathbb{P}\big(a_{n_k} < d\big) + \epsilon < 1 - \epsilon.$$

By independence of $(b_k)$

$$\mathbb{P}\big(b_k \geq \frac{2d}{3} \text{ for infinitely many } k\big) = 1.$$

Thus,

$$\mathbb{P}\big(a_{n_k} \geq \frac{d}{3} \text{ for infinitely many } k\big) > 1 - \epsilon.$$

**Assume the lemma.** $K = K(\bar{B}) =$ the continuum determined by $\bar{B} = (B_n)$

Suppose, for some continua $L^1$ and $L^2$,

$$K = L^1 \cup L^2.$$

Aim: $K = L^1$ or $K = L^2$.

For some intervals $J_n^1, J_n^2$,

$$I_n = J_n^1 \cup J_n^2,\ L^1 = \varprojlim_n (J_n^1,\ B_n \restriction J_{n+1}^1),\ L^2 = \varprojlim_n (J_n^2,\ B_n \restriction J_{n+1}^2).$$

From the lemma, there are $c > 0$ and $X \subseteq \mathbb{N}$ infinite

$$\max I_n > c,\ \min I_n < -c,\quad \text{for all } n \in X.$$

Also, for all $n$,

$$\{\min I_n,\ 0,\ \max I_n\} \subseteq J_n^1 \cup J_n^2.$$

We can assume there is $Y \subseteq X$ infinite such that

$$\{0, \max I_n\} \subseteq J_n^1, \text{ for all } n \in Y.$$

So

$$[0, c] \subseteq J_n^1, \text{ for all } n \in Y.$$

Then, for each $m$,

$$I_m = \lim_{n>m,\, n\in Y} \big(B_m \circ \cdots \circ B_n\big)([0, c]) \subseteq \overline{\bigcup_n \big(B_m \circ \cdots \circ B_n\big)(J_n^1)} = J_m^1$$

It follows that

$$K = L^1.$$

# Comments

# Wiener-type measure on continua

$\mathcal{C}(\mathbb{R}^{\mathbb{N}})$ = all subcontinua of $\mathbb{R}^{\mathbb{N}}$ with the Hausdorff metric

$\mathcal{C}(\mathbb{R}^{\mathbb{N}})$ is a closed subset of $\mathcal{K}(\mathbb{R}^{\mathbb{N}})$, so it is a Polish space.

By Theorem, the following set has measure 1:

$$\Omega_0 = \{\omega \in \Omega \mid B_n(\omega, \cdot) \text{ is continuous for each } n \in \mathbb{N}, \text{ and } \bar{B}(\omega) \text{ determines a continuum}\}$$

Consider the following function defined on $\Omega_0$:

$$\varphi \colon \Omega_0 \to \mathcal{C}(\mathbb{R}^{\mathbb{N}})$$
$$\varphi(\omega) = K\big(\bar{B}(\omega)\big).$$

**Proposition**

*The function $\varphi$ is measurable.*

The function $\varphi$ transfers the probability measure on $\Omega$ to $\mathcal{C}(\mathbb{R}^{\mathbb{N}})$.

$\beta =$ the transferred measure $=$ a **Wiener-type measure** on $\mathcal{C}(\mathbb{R}^{\mathbb{N}})$

Theorem is equivalent to asserting that the set

$$\{K \in \mathcal{C}(\mathbb{R}^{\mathbb{N}}) \mid K \text{ is a non-degenerate chainable indecomposable continuum}\}$$

is of full $\beta$ measure.

# The category context

$C_s([0,1]) =$ all continuous surjections from $[0,1]$ to itself

$C_s([0,1]) =$ is a Polish space with the uniform convergence topology.

$C_s([0,1])^{\mathbb{N}}$ also is a Polish space.

**Block, Keesling, Uspenskij** (2000): The set of $(f_n) \in C_s([0,1])^{\mathbb{N}}$ such that

$$\varprojlim_n ([0,1], f_n) \text{ is a non-degenerate indecomposable continuum}$$

is **comeager** in $C_s([0,1])^{\mathbb{N}}$. In fact, the same holds for

$$\varprojlim_n ([0,1], f_n) \text{ is non-degenerate, hereditarily indecomposable,}$$

that is, $\varprojlim_n ([0,1], f_n)$ is homeomorphic to the pseudoarc.

**Theorem (Casse–Curien, 2021)**

*$K(\bar{B})$ is* **not** *hereditarily indecomposable almost surely, so it is not homeomorphic to the pseudoarc.*

# Questions

(1) Is the measure $\beta$ in some sense canonical?

(2) What is the structure (almost surely) of the space of all subcontinua of the Polish space

$$\varprojlim_n (\mathbb{R}, B_n),$$

for a sequence $(B_n)$ of independent two-sided Brownian motions?